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# GRAPHS OF FUNCTIONS, REGULAR SETS AND S-STRAIGHT SETS

## Abstract

A subset  $E$  of  $\mathbb{R}^p$  is  $s$ -straight if  $E$  has finite Hausdorff  $s$ -dimensional outer measure which equals its Method I  $s$ -outer measure. The graph of a continuously differentiable function is shown to be the countable union of closed 1-straight sets together with a set of Hausdorff 1-measure zero. This result is extended to the graphs of absolutely continuous functions and to regular sets.

## 1 Introduction

In [?], Foran introduced the notion of  $s$ -straight and proposed a subset  $E$  of the unit circle such that  $E$  is 1-straight and has positive measure. A detailed analysis of this proposed set is given in [?] together with other examples and results for 1-straight sets.

Given a nonempty bounded subset  $B$  of  $\mathbb{R}^p$ , define  $\text{diam}(B) = \sup\{d(x, y) : x, y \in B\}$  where  $d(x, y)$  denotes the usual distance function in  $\mathbb{R}^p$ . Define  $\text{diam}(\emptyset) = 0$ . We write  $\text{diam}^s(E)$  in place of  $[\text{diam}(E)]^s$ .

**Definition 1.1** *Let  $E$  be a subset of  $\mathbb{R}^p$  and  $s > 0$ . Given  $\infty \geq \delta > 0$ , define*

$$\overline{m}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}^s(E_i) : E = \bigcup_{i=1}^{\infty} E_i, \text{diam}(E_i) < \delta \text{ for } i = 1, 2, \dots \right\}.$$

Set  $\overline{m}^s(E) = \sup_{\delta > 0} \overline{m}_\delta^s(E)$  and set  $\overline{m}_I^s(E) = \overline{m}_\infty^s(E)$ .

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The outer measure  $\overline{m}_I^s$  is known as a Method I outer measure. The outer measure  $\overline{m}^s$  is a metric outer measure on  $\mathbb{R}^p$ . Hence, every closed subset of  $\mathbb{R}^p$  is  $\overline{m}^s$ -measurable. See [?, pp. 132–144] for details. If  $E$  is an  $\overline{m}^s$ -measurable subset of  $\mathbb{R}^p$ , then we write  $\mathcal{H}^s(E)$  in place of  $\overline{m}^s(E)$ . If  $(A_n)_{n=1}^\infty$  is a decreasing sequence of compact subsets of  $\mathbb{R}^p$  and if  $\delta > 0$ , then  $\overline{m}_\delta^s(\bigcap_{n=1}^\infty A_n) = \lim_{n \rightarrow \infty} \overline{m}_\delta^s(A_n)$ . In [?], Foran defines s-straight and proves Theorem 1.3, providing a useful equivalent formulation of s-straight.

**Definition 1.2** *Given a subset  $E$  of  $\mathbb{R}^p$  and  $s > 0$ , we say  $E$  is s-straight if  $\overline{m}^s(E) < \infty$  and  $\overline{m}^s(E) = \overline{m}_I^s(E)$ .*

**Theorem 1.3** *Let  $E$  be a subset of  $\mathbb{R}^p$  with  $\overline{m}^s(E) < \infty$ . Then  $E$  is s-straight if and only if  $\overline{m}^s(E \cap K) \leq \text{diam}^s(E \cap K)$  for each compact subset  $K$  of  $\mathbb{R}^p$ .*

## 2 Graphs of Functions and S-Straight Sets

**Definition 2.1** *Let  $A_1, \dots, A_n$  be subsets of  $\mathbb{R}^p$ . We say that  $A_1, \dots, A_n$  are s-aligned if  $\text{diam}^s(B) \geq \sum_{i=1}^n \text{diam}^s(B \cap A_i)$  for each bounded subset  $B$  of  $A_1 \cup \dots \cup A_n$ .*

**Example.** If  $\text{diam}(A \cup B) \geq \text{diam}(A) + \text{diam}(B)$ , then  $A$  and  $B$  may not be 1-aligned. Let  $A = \{(x, \sin x) : 0 \leq x \leq \pi\}$  and  $B = \{(x, \sin x) : -\pi \leq x \leq 0\}$ . Then  $\text{diam}(A \cup B) = \text{diam}(A) + \text{diam}(B)$ . If  $A_1 = \{(x, \sin x) : 0 \leq x \leq \pi/2\}$ , then  $\text{diam}(A_1 \cup B) < \text{diam}(A_1) + \text{diam}(B)$ . Hence,  $A$  and  $B$  are not 1-aligned.

The motivation for the definition of s-aligned arises from the following result.

**Proposition 2.2** *Let  $A_1, \dots, A_n$  be s-aligned subsets of  $\mathbb{R}^p$ . If  $A_1, \dots, A_n$  are s-straight, then  $A_1 \cup \dots \cup A_n$  is s-straight.*

**Proof.** Let  $K$  be a compact set in  $\mathbb{R}^p$ . Then  $\overline{m}^s[K \cap (A_1 \cup \dots \cup A_n)] \leq \sum_{i=1}^n \overline{m}^s(K \cap A_i) \leq \sum_{i=1}^n \text{diam}^s(K \cap A_i) \leq \text{diam}^s[K \cap (A_1 \cup \dots \cup A_n)]$ .

Suppose  $A$  and  $B$  are s-aligned subsets of  $\mathbb{R}^p$ . Let  $\overline{A}$  and  $\overline{B}$  denote the closure of  $A$  and  $B$ , respectively. Then  $\overline{A} \cap \overline{B}$  contains at most one point. Hence,  $\overline{m}^s(A_1 \cup B_1) = \overline{m}^s(A_1) + \overline{m}^s(B_1)$  for each  $A_1 \subseteq \overline{A}$  and  $B_1 \subseteq \overline{B}$ .

**Theorem 2.3** *Let  $(q_n)_{n=1}^\infty$  be a sequence of positive integers. Set  $Q^n = \{(i_1, \dots, i_n) : 1 \leq i_1 \leq q_1, \dots, 1 \leq i_n \leq q_n\}$  for  $n = 1, 2, \dots$ . Let  $\{A_\beta : \beta \in Q^n \text{ for some } n \geq 1\}$  be a family of compact subsets of  $\mathbb{R}^p$  such that  $A_1, \dots, A_{q_1}$  are s-aligned and such that*

- (a)  $A_{(\beta,1)}, \dots, A_{(\beta,q_n)}$  are s-aligned if  $\beta \in Q^{n-1}$ ,
- (b)  $A_\beta \supseteq A_{(\beta,1)} \cup \dots \cup A_{(\beta,q_n)}$  if  $\beta \in Q^{n-1}$  and

(c)  $\lim_{n \rightarrow \infty} \max\{\text{diam}^s(A_\beta) : \beta \in Q^n\} = 0$ .  
 Let  $P_n = \bigcup\{A_\beta : \beta \in Q^n\}$ . Then  $\bigcap_{n=1}^\infty P_n$  is s-straight.

**Proof.** If  $E \subseteq P_n$ , then  $\text{diam}^s(E) \geq \sum_{\beta \in Q^n} \text{diam}^s(E \cap A_\beta)$ . Let  $P = \bigcap_{n=1}^\infty P_n$ . The set  $P$  is closed and  $P_n \supseteq P_{n+1}$  for  $n \geq 1$ . Let  $K$  be a compact subset of  $\mathbb{R}^p$ . It suffices to show that  $\text{diam}^s(P \cap K) \geq \overline{m}_\delta^s(P \cap K)$  for each  $\delta > 0$ . Let  $\delta > 0$ . Choose  $\ell$  such that  $\text{diam}^s(A_\beta) < \delta$  for each  $\beta \in Q^\ell$ . Then for each  $n \geq \ell$

$$\text{diam}^s(P_n \cap K) \geq \sum_{\beta \in Q^n} \text{diam}^s(A_\beta \cap K) \geq \overline{m}_\delta^s(P_n \cap K).$$

Hence,  $\lim_{n \rightarrow \infty} \text{diam}^s(P_n \cap K) \geq \lim_{n \rightarrow \infty} \overline{m}_\delta^s(P_n \cap K) \geq \overline{m}_\delta^s(P \cap K)$ . Since  $P$  is closed,  $\text{diam}^s(P \cap K) = \lim_{n \rightarrow \infty} \text{diam}^s(P_n \cap K)$ . Thus,  $\text{diam}^s(P \cap K) \geq \overline{m}_\delta^s(P \cap K)$ . It follows that  $P$  is s-straight.

**Lemma 2.4** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuously differentiable with  $0 \leq m \leq f'(t) \leq M$ . If  $I$  is a subinterval of  $[a, b]$ , let  $F(I) = \{(x, f(x)) : x \in I\}$ . Suppose  $0 < w < 1$  satisfies  $(1+w)\sqrt{1+m^2} \geq 2w\sqrt{1+M^2}$ . Then for each positive integer  $n$ , there exist disjoint subintervals  $I_1, \dots, I_n$  of  $[a, b]$  each of length  $w(b-a)/n$  such that  $\mathcal{H}^1[\bigcup_{k=1}^n F(I_k)] \geq w^2 \mathcal{H}^1[F([a, b])]$  and  $F(I_1), \dots, F(I_n)$  are 1-aligned.*

**Proof.** It suffices to consider the case where  $[a, b] = [0, 1]$ . Let  $F(s, t) = \{(x, f(x)) : s \leq x \leq t\}$  if  $0 \leq s \leq t \leq 1$ . Assume  $(1+w)\sqrt{1+m^2} \geq 2w\sqrt{1+M^2}$  where  $0 < w < 1$ . Set  $\alpha = \arctan m$  and  $\beta = \arctan M$ . Then  $\tan \alpha \leq f' \leq \tan \beta$  and  $(1+w)\sec \alpha \geq 2w\sec \beta$ . If  $0 \leq s \leq t \leq 1$ , then  $(t-s)\sec \alpha \leq \mathcal{H}^1[F(s, t)] \leq (t-s)\sec \beta$ . Let  $n$  be a positive integer. Set  $\delta = (1-w)/(2n)$ . Define  $a_k = (k-1)/n + \delta$ ,  $b_k = k/n - \delta$  and  $I_k = [a_k, b_k]$  for  $k = 1, \dots, n$ . Then  $\mathcal{H}^1[\bigcup_{k=1}^n F(I_k)] = \sum_{k=1}^n \mathcal{H}^1[F(I_k)] \geq \sum_{k=1}^n |I_k| \sec \alpha = w \sec \alpha$  where  $|I_k|$  denotes the length of  $I_k$ . Hence

$$\mathcal{H}^1\left[\bigcup_{k=1}^n F(I_k)\right] \geq w \frac{\sec \alpha}{\sec \beta} \mathcal{H}^1[F(0, 1)] \geq \frac{2w^2}{1+w} \mathcal{H}^1[F(0, 1)] \geq w^2 \mathcal{H}^1[F(0, 1)].$$

Since  $f$  is increasing, to show that  $F(I_1), \dots, F(I_n)$  are 1-aligned, it suffices to show that  $\text{diam}[F(s, t)] \geq \sum_{k=1}^n \text{diam}[F([s, t] \cap I_k)]$  if  $0 \leq s < t \leq 1$ . If  $s, t$  belong to a single  $I_k$ , this inequality is clear. Let  $a_j \leq s < b_j$  and let  $a_k < t \leq b_k$  where  $j < k$ . Set

$$\Phi(s, t) = \text{diam}[F(s, t)] - \text{diam}[F(s, b_j)] - \text{diam}[F(a_k, t)] - \sum_{i=j+1}^{k-1} \text{diam}[F(a_i, b_i)].$$

To show that  $F(I_1), \dots, F(I_n)$  are 1-aligned, it suffices to show that  $\Phi(s, t) \geq 0$ . Set  $m = k + 1 - j \geq 2$ . We have

$$\begin{aligned}
\Phi(s, t) &\geq (t - s) \sec \alpha - \left[ (b_j - s) + (t - a_k) + \sum_{i=j+1}^{k-1} (b_i - a_i) \right] \sec \beta \\
&\geq (b_k - a_j) \sec \alpha - \left[ (b_j - a_j) + (b_k - a_k) + \sum_{i=j+1}^{k-1} (b_i - a_i) \right] \sec \beta \\
&= \frac{m - (1 - w)}{n} \sec \alpha - \frac{mw}{n} \sec \beta \\
&= \frac{m}{n} \left[ \frac{m - 1 + w}{m} \sec \alpha - w \sec \beta \right] \\
&\geq \frac{m}{n} \left[ \frac{1 + w}{2} \sec \alpha - w \sec \beta \right].
\end{aligned}$$

Hence,  $\Phi(s, t) \geq 0$  and so  $F(I_1), \dots, F(I_n)$  are 1-aligned.

**Theorem 2.5** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuously differentiable with  $0 \leq f' \leq M$ . Let  $\Gamma = \{(x, f(x)) : a \leq x \leq b\}$ . Then there exists closed 1-straight subset  $P$  of  $\Gamma$  such that  $\mathcal{H}^1(P) \geq \mathcal{H}^1(\Gamma)/[4(1 + M^2)]$ .*

**Proof.** If  $I$  is a subinterval of  $[a, b]$ , set  $F(I) = \{(x, f(x)) : x \in I\}$ ,  $m(f', I) = \min\{f'(x) : x \in I\}$  and  $M(f', I) = \max\{f'(x) : x \in I\}$ . Set  $\beta = \arctan(M)$  and let  $w_1$  be the positive number such that  $1 + w_1 = 2w_1 \sec \beta$ , that is,  $w_1 = 1/(2 \sec \beta - 1)$ . Then  $w_1^2 > 1/[4(1 + M^2)]$ . Choose  $w_2, w_3, \dots$  in  $(0, 1)$  such that  $\prod_{k=1}^{\infty} w_k^2 = 1/[4(1 + M^2)]$ . By Lemma ?? and by uniform continuity of  $f'$  on  $[a, b]$ , we may choose an integer  $q_1 \geq 2$  and closed subintervals  $I_1, \dots, I_{q_1}$  of  $[a, b]$  each of length  $w_1(b - a)/q_1$  such that

$$(\text{step 1}) \left\{ \begin{array}{l} F(I_1), \dots, F(I_{q_1}) \text{ are 1-aligned,} \\ \mathcal{H}^1[\bigcup_{j=1}^{q_1} F(I_j)] \geq w_1^2 \mathcal{H}^1(\Gamma) \text{ and} \\ \frac{1 + w_2}{2w_2} \sqrt{\frac{1 + m^2(f', I_j)}{1 + M^2(f', I_j)}} \geq 1 \text{ for } j = 1, \dots, q_1 \end{array} \right.$$

Let  $P_1 = F(I_1) \cup \dots \cup F(I_{q_1})$ . Then  $\mathcal{H}^1(P_1) \geq w_1^2 \mathcal{H}^1(\Gamma)$ . By Lemma ?? and by uniform continuity of  $f'$  on  $[a, b]$ , we may choose an integer  $q_2 \geq 2$  and closed subintervals  $I_{i,1}, \dots, I_{i,q_2}$  of  $I_i$  each of length  $w_1 w_2 (b - a)/(q_1 q_2)$  for  $i = 1, \dots, q_1$  such that

$$(\text{step 2}) \left\{ \begin{array}{l} F(I_{i,1}), \dots, F(I_{i,q_2}) \text{ are 1-aligned for } i = 1, \dots, q_1 \\ \mathcal{H}^1[\bigcup_{j=1}^{q_2} F(I_{i,j})] \geq w_2^2 \mathcal{H}^1[F(I_i)] \text{ for } i = 1, \dots, q_1 \text{ and} \\ \frac{1 + w_3}{2w_3} \sqrt{\frac{1 + m^2(f', I_{i,j})}{1 + M^2(f', I_{i,j})}} \geq 1 \text{ for } i = 1, \dots, q_1 \text{ and } j = 1, \dots, q_2 \end{array} \right.$$

Let  $P_2 = \bigcup\{F(I_{i,j}) : 1 \leq i \leq q_1 \text{ and } 1 \leq j \leq q_2\}$ . Then  $\mathcal{H}^1(P_2) \geq w_1^2 w_2^2 \mathcal{H}^1(\Gamma)$ . Continuing this process, we obtain, in accordance with Theorem ??, a decreasing sequence  $(P_n)_{n=1}^\infty$  of compact subsets of  $\Gamma$  such that  $\mathcal{H}^1(P_n) \geq (\prod_{k=1}^n w_k^2) \mathcal{H}^1(\Gamma)$  for each  $n \geq 1$  and  $\bigcap_{n=1}^\infty P_n$  is 1-straight. Let  $P = \bigcap_{n=1}^\infty P_n$ . Then  $P$  is a closed 1-straight subset of  $\Gamma$  with  $\mathcal{H}^1(P) \geq \mathcal{H}^1(\Gamma)/[4(1 + M^2)]$ .

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and let  $\Gamma$  be the graph of  $f$ . Assume that  $\mathcal{H}^1(\Gamma)$  is finite. If each closed subset of  $\Gamma$  with positive  $\mathcal{H}^1$  measure contains a closed 1-straight subset of positive  $\mathcal{H}^1$  measure, then  $\Gamma$  is the countable union of closed 1-straight subsets together with a set of  $\mathcal{H}^1$  measure zero since the measure  $\mathcal{H}^1$  is regular. See [?] for a similar result.

**Corollary 2.6** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuously differentiable. Then the graph of  $f$  is the countable union of closed 1-straight sets together with a set of  $\mathcal{H}^1$  measure zero.*

**Proof.** If  $I$  is a subinterval of  $[a, b]$ , let  $F(I) = \{(x, f(x)) : x \in I\}$ . Choose closed subintervals  $I_1, \dots, I_n$  of  $[a, b]$  that cover  $[a, b]$  such that  $f' \geq 0$ ,  $f' \leq 0$  or  $|f'| \leq \sqrt{3}/3$  on  $I_k$  for each  $k = 1, \dots, n$ . If  $|f'| \leq \sqrt{3}/3$  on  $I_j$ , then  $F(I_j)$  rotated by  $30^\circ$  coincides with the graph of a continuously differentiable function  $g$  with  $g' \geq 0$ . It follows from Theorem ?? that each  $F(I_k)$  is the countable union of closed 1-straight subsets together with a subset of  $\mathcal{H}^1$  measure zero and so likewise for the graph of  $f$ .

**Theorem 2.7** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be absolutely continuous. Then the graph of  $f$  is the countable union of closed 1-straight sets together with a set of  $\mathcal{H}^1$  measure zero.*

**Proof.** Let  $\Gamma_f = \{(x, f(x)) : a \leq x \leq b\}$ . Let  $\mu$  denote Lebesgue measure on  $[a, b]$ . Let  $E$  be a closed subset of  $\Gamma_f$  with positive  $\mathcal{H}^1$  measure. Set  $B = \{x \in [a, b] : (x, f(x)) \in E\}$ . Since  $f$  is absolutely continuous,  $B$  is a compact subset of  $[a, b]$  with  $\mu(B) > 0$ . But  $f$  is differentiable almost everywhere and  $f'$  is  $\mu$ -measurable. By a theorem due to Federer, see [?, Theorem 3.1.15] and [?, p. 1160], there exists a compact subset  $K$  of  $B$  and a continuously differentiable function  $g$  on  $[a, b]$  such that  $g = f$  on  $K$  and  $\mu(K) > 0$ . Let  $\Gamma_g(K) = \{(x, g(x)) : x \in K\}$ . Then  $\Gamma_g(K)$  is a compact subset of the graph of  $g$  with positive  $\mathcal{H}^1$  measure. By Corollary ??, there exists a closed 1-straight subset  $P$  of the graph of  $g$  with  $\mathcal{H}^1(P \cap \Gamma_g(K)) > 0$ . Since  $P$  is 1-straight,  $P \cap \Gamma_g(K)$  is a closed 1-straight subset of  $E$ . Hence, the graph of  $f$  is the countable union of closed 1-straight subsets together with a set of  $\mathcal{H}^1$  measure zero.

**Corollary 2.8** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and increasing. Then the graph of  $f$  is the countable union of closed 1-straight sets together with a set of  $\mathcal{H}^1$  measure zero.*

**Proof.** The graph of  $f$  rotated clockwise by  $45^\circ$  coincides with the graph of a continuous Lipschitz function, which is absolutely continuous.

### 3 Regular Sets and S-Straight Sets

If  $E \subset \mathbb{R}^p$ , we say  $E$  is an s-set if  $E$  is  $\overline{m}^s$ -measurable and  $0 < \mathcal{H}^s(E) < \infty$ . If  $x \in \mathbb{R}^p$ , let  $B_r(x)$  denote the closed ball of radius  $r > 0$  centered at  $x$ . We next recall some definitions concerning density, regular sets and contingents.

**Definition 3.1** [?, pp. 20-21]. We say  $x$  is a regular point of  $E$  if the  $\mathcal{H}^s$ -density of  $x$  with respect to  $E$ , defined as  $\lim_{r \rightarrow 0^+} [\mathcal{H}^s(E \cap B_r(x)) / (2r)^s]$ , exists and equals 1. A set  $E$  is called a regular set if almost every point of  $E$  is regular. Here we consider only the case  $s = 1$  and sets  $E \subseteq \mathbb{R}^2$ .

**Definition 3.2** [?, pp. 26-29]. A curve (or, Jordan curve)  $\Gamma$  is the image of a continuous one-to-one function  $\psi : [a, b] \rightarrow \mathbb{R}^p$ , where  $[a, b] \subseteq \mathbb{R}$  is a closed interval. In particular, a curve is not self-intersecting. If the length  $\mathcal{L}(\Gamma)$  of a curve  $\Gamma$  is defined in the usual way and if  $\mathcal{L}(\Gamma) < \infty$ , we say that  $\Gamma$  is a rectifiable curve. A 1-set contained in a countable union of rectifiable curves is called a  $Y$ -set [?, p. 33]. Let  $S(x, \theta, \varphi)$  be the closed one-way infinite cone with vertex  $x$  and axis in direction of angle  $\theta$  consisting of those points  $y$  such that the line segment between  $x$  and  $y$  makes an angle of at most  $\varphi$  with that axis.

**Definition 3.3** [?, p. 262]. Let  $E \subseteq \mathbb{R}^2$ . For any point  $x \in E$ , the direction  $\theta$  of a half-line originating at  $x$  is the angle made by that half-line with a fixed direction, usually that of the horizontal axis. Such a half-line is denoted by  $l(x, \theta)$ . A half-line containing a point  $y \neq x$  is denoted by  $\overrightarrow{xy}$ . A sequence  $\{l_n(x, \theta_n)\}_{n=1}^\infty$  of half-lines is said to converge to the half-line  $l(x, \theta)$  if  $\lim_{n \rightarrow \infty} \theta_n = \theta$ . A half-line  $l(x, \theta)$  is called an intermediate half-tangent of  $E$  at  $x$  if there exists a sequence of points  $\{x_n\}_{n=1}^\infty \subseteq E$ , with  $x_n \neq x$  for all  $n$  such that both  $\lim_{n \rightarrow \infty} x_n = x$ , and the sequence of half-lines  $\{\overrightarrow{x_n x}\}_{n=1}^\infty$  converges to  $l(x, \theta)$ . Finally, the contingent of  $E$  at  $x$ , denoted by  $\text{contg}_E(x)$ , is the set of all intermediate half-tangents of  $E$  at  $x$ . (If  $x_0$  is an isolated point, then  $x_0$  has no intermediate half-tangents, and  $\text{contg}_E(x_0) = \emptyset$ .)

The following Lemma is part of the so-called Fundamental Theorem on Contingents of Plane Sets, found in Saks.

**Lemma 3.4** [?, p. 264]. Let  $\theta$  be a fixed direction, and  $E \subseteq \mathbb{R}^2$  be such that for each  $x \in E$ , the set  $\text{contg}_E(x)$  contains no half-line of direction  $\theta$ . If  $\theta$  is the direction of the positive vertical axis, then  $E = \cup_{i=1}^\infty E_i$  such that for each

*i* the set  $E_i$  is the graph of a Lipschitz function  $f_i : B_i \rightarrow \mathbb{R}$  where  $B_i$  is a bounded subset of  $\mathbb{R}$  and  $\mathcal{H}^1(E_i) = \mathcal{L}(E_i) < \infty$ .

**Theorem 3.5** *Let  $E \subseteq \mathbb{R}^2$  be a regular 1-set. Then  $E$  is a countable union of 1-straight sets together with a set of  $\mathcal{H}^1$  measure zero.*

**Proof.** Besicovitch proved (see [?, p. 45]) that a regular 1-set  $E \subseteq \mathbb{R}^2$  consists of a  $Y$ -set together with a set of  $\mathcal{H}^1$ -measure zero. In his proof (see [?, p. 32]) that a rectifiable curve  $\Gamma$  in  $\mathbb{R}^2$  has a tangent at almost all of its points, he proved in particular that for almost all  $x \in \Gamma$  there exists a direction  $\theta$  such that for suitable  $\varphi, \rho > 0$  it follows that  $\Gamma \cap [B_\rho(x) \setminus (S(x, \theta, \varphi) \cup S(x, \theta + \pi, \varphi))] = \emptyset$ . So at almost all points  $x$  of a rectifiable curve  $\Gamma$ ,  $\text{contg}_\Gamma(x) \neq \mathbb{R}^2$ . As in the proof of the Fundamental Theorem on Contingents of Plane Sets [?, p. 267], let  $\{\theta_n\}$  be a countable everywhere dense set of directions in  $\mathbb{R}^2$ . Let  $\Gamma_n$  be the set of points of  $\Gamma$  at which  $\text{contg}_\Gamma(x)$  does not contain the half-line of direction  $\theta_n$ . Then since  $\{\theta_n\}$  is dense, it is clear that  $\Gamma = \cup_{n=1}^\infty \Gamma_n$ , otherwise there exist points  $x \in \Gamma$  such that  $\text{contg}_\Gamma(x) = \mathbb{R}^2$ . By Lemma ??, with respect to a line of direction  $\theta_n + \frac{\pi}{2}$ , each  $\Gamma_n$  is the countable union of finite length graphs of Lipschitz functions on bounded domain sets. Thus  $\Gamma$  itself equals such a union. It is well-known, as in [?, p. 264], that any such Lipschitz function can be extended to be Lipschitz on the smallest closed interval containing its bounded domain. Since a Lipschitz function is absolutely continuous, by Theorem ?? it follows that the graph of each such extended Lipschitz function is a countable union of 1-straight sets. Since subsets of 1-straight sets are 1-straight and since translations and rotations of 1-straight sets are 1-straight (see [?]), the subset  $\Gamma$  of the countable union of the graphs of these Lipschitz functions is also a countable union of 1-straight sets. Since this is true for each rectifiable curve  $\Gamma$  in  $\mathbb{R}^2$ , it follows that  $E$  is therefore a countable union of 1-straight sets together with a set of  $\mathcal{H}^1$ -measure zero.

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