Halley Chapman, 6th grade teacher at Southwest Early College Campus, was surprised that a standard approximation did not uniquely determine where the decimal point should go in the product of two decimals. This lead me to the following:

**Theorem:** As notation, let \( n, m \in \{0, 1, 2, \ldots \} \), \( \delta, \varepsilon \) satisfy \( 0 \leq \delta, \varepsilon \leq 1 \), and \( k \in \{1, 2, \ldots \} \). Assume we have the following decimal multiplication question, with upper bound and lower bound approximations as shown:

\[
\begin{align*}
&\begin{cases}
n \leq n + \varepsilon \leq n + 1 \\
m \leq m + \delta \leq m + 1 \\
\text{Multiplying vertically}
\end{cases}
\end{align*}
\]

\[nm \leq (n + \varepsilon)(m + \delta) \leq (n + 1)(m + 1).
\]

So, the "correct product" is:

\[(n + \varepsilon)(m + \delta)\]

**Part (1)**

We have \(10^{-k}\) times the correct product lying between the approximations, meaning

\[nm \leq 10^{-k}(n + \varepsilon)(m + \delta) \leq (n + 1)(m + 1)\]

\[\iff \quad \begin{cases} n = 0 \quad \text{or} \quad m = 0 \end{cases}.\]

**Part (2)**

We have \(10^{j}\) times the correct product lying between the approximations (for all \( j = 1, 2, \ldots, k \)), meaning

\[nm \leq 10^{j}(n + \varepsilon)(m + \delta) \leq (n + 1)(m + 1)\]

\[\iff \quad \begin{cases} n = 0 \quad \text{and} \quad 0 \leq \varepsilon \leq \frac{1}{10^{k}} \left(\frac{n+1}{m+\delta}\right) \\
or \quad m = 0 \quad \text{and} \quad 0 \leq \delta \leq \frac{1}{10^{k}} \left(\frac{n+1}{n+\varepsilon}\right)\end{cases}.\]
PROOF:
We note the correct product for reference:
\[(n + \varepsilon)(m + \delta) = nm + n\delta + m\varepsilon + \delta\varepsilon\]

Part (1): If \(10^{-k}\) times the correct product is to lie between the two approximations for \(k \geq 1\), it is clear that we need only check that the \underline{left} inequality holds, meaning that
\[nm \leq 10^{-k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \quad (*)\]

\[\iff\] If \(n = 0\), then inequality (*) becomes
\[0 \leq 10^{-k}(m\varepsilon + \delta\varepsilon)\]
which is true. Likewise, if \(m = 0\) the associated inequality (*)
\[0 \leq 10^{-k}(n\delta + \delta\varepsilon)\]
is similarly true.

\[\implies\] By contrapositive, now suppose both \(n \neq 0\) and \(m \neq 0\), meaning both \(n \geq 1\) and \(m \geq 1\). We then have
\[n\delta \leq n \cdot 1 \leq nm ,\]
\[m\varepsilon \leq m \cdot 1 \leq nm ,\] and
\[\delta\varepsilon \leq 1 \cdot 1 \leq nm .\]

Adding vertically, we get
\[n\delta + m\varepsilon + \delta\varepsilon \leq 3nm .\]
But then because \(k \geq 1\), we have \(10^{-k} \leq 10^{-1}\), so
\[10^{-k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq 10^{-1} \cdot 4nm < nm\]
which is the negation of inequality (*)

Part (2): If \(10^k\) times the correct product is to lie between the two approximations (and likewise for each \(10^j\) times the correct product for all \(j = 1,2,\ldots,k\), it is clear that we need only check that the \underline{right} inequality holds, meaning that
\[10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq (n + 1)(m + 1) \quad (**)\]
If $n = 0$ and $0 \leq \varepsilon \leq \frac{1}{10^k} \left( \frac{m + 1}{m + \delta} \right)$, then because $n = 0$, inequality (**) becomes

$$10^k(m\varepsilon + \delta\varepsilon) \leq m + 1.$$  

The condition on $\varepsilon$ thus implies

$$10^k(m\varepsilon + \delta\varepsilon)$$

$$= (m + \delta)10^k\varepsilon$$

$$\leq (m + \delta)10^k \left[ \frac{1}{10^k} \left( \frac{m + 1}{m + \delta} \right) \right]$$

$$= m + 1$$

so inequality (**) holds, as desired. Similarly, if

$$m = 0 \text{ and } 0 \leq \delta \leq \frac{1}{10^k} \left( \frac{n + 1}{n + \varepsilon} \right),$$

then inequality (**) becomes

$$10^k(n\delta + \delta\varepsilon) \leq n + 1,$$

and the condition on $\delta$ thus implies

$$10^k(n\delta + \delta\varepsilon)$$

$$= (n + \varepsilon)10^k\delta$$

$$\leq (n + \varepsilon)10^k \left[ \frac{1}{10^k} \left( \frac{n + 1}{n + \varepsilon} \right) \right]$$

$$= n + 1$$

so again inequality (**) holds, as desired.

[ $\Rightarrow$ ] By contrapositive, now suppose both

$$n \neq 0 \text{ or } \varepsilon > \frac{1}{10^k} \left( \frac{m + 1}{m + \delta} \right)$$

and

$$m \neq 0 \text{ or } \delta > \frac{1}{10^k} \left( \frac{n + 1}{n + \varepsilon} \right)$$

There are 4 cases to examine.
**Case 1:** $n \neq 0$ and $m \neq 0$.
Since both $n \geq 1$ and $m \geq 1$, $\delta, \varepsilon$ are non-negative, and $k \geq 1$ we see that
\[
10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) \geq 10nm
\]
\[
= nm + nm + nm + nm + 6nm
\]
\[
> nm + n + m + 1
\]
\[
= (n + 1)(m + 1)
\]
which is the negation of inequality $(\ast \ast)$, as desired.

**Case 2:** $n \neq 0$ (hence $n \geq 1$) and $\delta > \frac{1}{10^k} \left( \frac{n+1}{n+\varepsilon} \right)$.
Since $n \geq 1$, $\delta, \varepsilon$ are non-negative, and $k \geq 1$, using the given $\delta$ inequality we have
\[
10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) = 10^knm + 10^k \delta(n + \varepsilon) + 10^k m\varepsilon
\]
\[
\geq 10nm + 10^k \left[ \frac{1}{10^k} \left( \frac{n + 1}{n + \varepsilon} \right) \right](n + \varepsilon)
\]
\[
= 8nm + nm + nm + n + 1
\]
\[
> nm + m + n + 1
\]
\[
= (n + 1)(m + 1)
\]
which is the negation of inequality $(\ast \ast)$, as desired.

**Case 3:** $\varepsilon > \frac{1}{10^k} \left( \frac{m+1}{m+\delta} \right)$ and $m \neq 0$.
The proof here is exactly analogous to that of Case 2.

**Case 4:** $\varepsilon > \frac{1}{10^k} \left( \frac{m+1}{m+\delta} \right)$ and $\delta > \frac{1}{10^k} \left( \frac{n+1}{n+\varepsilon} \right)$.
Observe that here we can also assume that $n = m = 0$, since if not, we reduce to one of Cases 1, 2, or 3. So, inequality $(\ast \ast)$ reduces from
\[
10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq (n + 1)(m + 1)
\]
to
\[
10^k\delta\varepsilon \leq 1
\]
Also, our two given conditions reduce to
\[
\varepsilon > \frac{1}{10^k}\frac{1}{\delta} \quad \text{and} \quad \delta > \frac{1}{10^k}\frac{1}{\varepsilon}
\]
meaning the one condition:
\[
10^k\delta\varepsilon > 1
\]
which is once again the negation of inequality $(\ast \ast)$, as desired. ■