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Halley Chapman, 6th grade teacher at Southwest Early College Campus, was surprised that a standard approximation did not uniquely determine where the decimal point should go in the product of two decimals. This lead me to the following:

Theorem: As notation, let $n, m \in \{0, 1, 2, \ldots\}$, δ, ε satisfy $0 \le \delta, \varepsilon \le 1$, and $k \in \{1, 2, \ldots\}$. Assume we have the following decimal multiplication question, with upper bound and lower bound approximations as shown:

$$\left[\begin{array}{c} n \le n+\varepsilon \le n+1 \\ m \le m+\delta \le m+1 \\ \text{Multiplying vertically} \\ nm \le (n+\varepsilon)(m+\delta) \le (n+1)(m+1)\ . \end{array}\right]$$

So, the "correct product" is: $(n+\varepsilon)(m+\delta)$

Part (1)

$$\left[\begin{array}{c} \text{We have } 10^{-k} \text{ times the correct product} \\ \text{lying between the approximations,} \\ \text{meaning} \\ nm \le 10^{-k}(n+\varepsilon)(m+\delta) \le (n+1)(m+1) \end{array}\right]$$

$$\Leftrightarrow$$

$$[\ n = 0 \quad \text{or} \quad m = 0\ ].$$

Part (2)

$$\left[\begin{array}{c} \text{We have } 10^{j} \text{ times the correct product} \\ \text{lying between the approximations} \\ (\text{ for all } j = 1, 2, \ldots, k\ ), \\ \text{meaning} \\ nm \le 10^{j}(n+\varepsilon)(m+\delta) \le (n+1)(m+1) \end{array}\right]$$

$$\Leftrightarrow$$

$$\left[\begin{array}{c} n = 0 \ \text{ and } \ 0 \le \varepsilon \le \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right) \\ \text{or} \\ m = 0 \ \text{ and } \ 0 \le \delta \le \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) \end{array}\right]$$

PROOF:

We note the correct product for reference:

$$(n+\varepsilon)(m+\delta) = nm + n\delta + m\varepsilon + \delta\varepsilon$$

Part (1): If 10^{-k} times the correct product is to lie between the two approximations for $k \geq 1$, it is clear that we need only check that the <u>left</u> inequality holds, meaning that

$$nm \leq 10^{-k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \qquad (*)$$

[$\Leftarrow$] If $n = 0$, then inequality $(*)$ becomes

$$0 \leq 10^{-k}(m\varepsilon + \delta\varepsilon)$$

which is true. Likewise, if $m = 0$ the associated inequality $(*)$

$$0 \leq 10^{-k}(n\delta + \delta\varepsilon)$$

is similarly true.

[$\Rightarrow$] By contrapositive, now suppose both $n \neq 0$ and $m \neq 0$, meaning both $n \geq 1$ and $m \geq 1$. We then have

$$n\delta \leq n \cdot 1 \leq nm \text{ ,}$$

$$m\varepsilon \leq m \cdot 1 \leq nm \text{ , and}$$

$$\delta\varepsilon \leq 1 \cdot 1 \leq nm \text{ .}$$

Adding vertically, we get

$$n\delta + m\varepsilon + \delta\varepsilon \leq 3nm \text{ .}$$

But then because $k \geq 1$, we have $10^{-k} \leq 10^{-1}$, so

$$10^{-k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq 10^{-1} \cdot 4nm < nm$$

which is the negation of inequality $(*)$.

Part (2): If 10^{k} times the correct product is to lie between the two approximations (and likewise for each 10^{j} times the correct product for all $j = 1, 2, \ldots, k$), it is clear that we need only check that the <u>right</u> inequality holds, meaning that

$$10^{k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq (n+1)(m+1) \qquad (**)$$

[$\Leftarrow$] If

$$n = 0 \;\text{ and }\; 0 \le \varepsilon \le \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right),$$

then because $n = 0$, inequality $(**)$ becomes

$$10^k(m\varepsilon + \delta\varepsilon) \le m + 1 \,.$$

The condition on ε thus implies

$$\begin{aligned} &10^k(m\varepsilon + \delta\varepsilon) \\ &= (m+\delta)10^k\varepsilon \\ &\le (m+\delta)10^k\left[\frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)\right] \\ &= m+1 \end{aligned}$$

so inequality $(**)$ holds, as desired. Similarly, if

$$m = 0 \;\text{ and }\; 0 \le \delta \le \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right),$$

then inequality $(**)$ becomes

$$10^k(n\delta + \delta\varepsilon) \le n + 1,$$

and the condition on δ thus implies

$$\begin{aligned} &10^k(n\delta + \delta\varepsilon) \\ &= (n+\varepsilon)10^k\delta \\ &\le (n+\varepsilon)10^k\left[\frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)\right] \\ &= n+1 \end{aligned}$$

so again inequality $(**)$ holds, as desired.

[$\Rightarrow$] By contrapositive, now suppose both

$$\left[\begin{array}{c} n \ne 0 \;\text{ or }\; \varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right) \\ \text{and} \\ m \ne 0 \;\text{ or }\; \delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) \end{array}\right]$$

There are 4 cases to examine.

Case 1: $n \neq 0$ and $m \neq 0$.

Since both $n \geq 1$ and $m \geq 1$, δ, ε are non-negative, and $k \geq 1$ we see that

$$\begin{aligned} 10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) &\geq 10nm \\ &= nm + nm + nm + nm + 6nm \\ &> nm + n + m + 1 \\ &= (n+1)(m+1)\ , \end{aligned}$$

which is the negation of inequality $(* *)$, as desired.

Case 2: $n \neq 0$ (hence $n \geq 1$) and $\delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)$.

Since $n \geq 1$, δ, ε are non-negative, and $k \geq 1$, using the given δ inequality we have

$$\begin{aligned} 10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) &= 10^k nm + 10^k\delta(n+\varepsilon) + 10^k m\varepsilon \\ &\geq 10nm + 10^k\left[\frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)\right](n+\varepsilon) \\ &= 8nm + nm + nm + n + 1 \\ &> nm + m + n + 1 \\ &= (n+1)(m+1)\ , \end{aligned}$$

which is the negation of inequality $(* *)$, as desired.

Case 3: $\varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)$ and $m \neq 0$.

The proof here is exactly analogous to that of **Case 2**.

Case 4: $\varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)$ and $\delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)$.

Observe that here we can also assume that $n = m = 0$, since if not, we reduce to one of **Cases 1, 2, or 3**. So, inequality $(* *)$ reduces from

$$10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) \leq (n+1)(m+1)$$

to

$$10^k\delta\varepsilon \leq 1\ .$$

Also, our two given conditions reduce to

$$\varepsilon > \frac{1}{10^k}\frac{1}{\delta} \quad \text{and} \quad \delta > \frac{1}{10^k}\frac{1}{\varepsilon}\ ,$$

meaning the one condition:

$$10^k\delta\varepsilon > 1$$

which is once again the negation of inequality $(* *)$, as desired. ■