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Halley Chapman, Grade 6 Mathematics teacher, was surprised that a standard approximation technique, often used in the multiplication of two decimals, did not uniquely determine where the decimal point should go in the product. Here is an example like the one that surprised her:

$$
\begin{aligned}
4 &\leq 4.25 \leq 5 \\
0 &\leq .32 \leq 1 \\
&\text{Multiplying vertically,} \\
0 &\leq 1.3600 \leq 5
\end{aligned}
$$

The correct product of 4.25 and .32 is 1.3600 as shown. (The two trailing zeros just reflect the standard algorithm used for the product.) But, if a student did not know where to place the decimal point in the product, the integer lower bound and integer upper bound products of 0 and 5 are intended to force only one possible choice for that decimal point placement between those bounds. However, in addition to the correct answer, we see that .13600 also works here. So, we have two possibilities, which is what elicited Halley's surprise.

Question: For what choices of decimal numbers to multiply does this upper bound - lower bound approximation process fail to limit the choice of decimal point placement in the product to a single possibility?

Answer: As notation, let $n, m \in \{0, 1, 2, ...\}$, $k \in \{1, 2, ...\}$, and let δ, ε satisfy $0 \leq \delta, \varepsilon \leq 1$. Assume we have the following decimal multiplication question, with upper bound and lower bound approximations as shown:

$$
\left[
\begin{array}{c}
n \leq n+\varepsilon \leq n+1 \\
m \leq m+\delta \leq m+1 \\
\text{Multiplying vertically,} \\
nm \leq (n+\varepsilon)(m+\delta) \leq (n+1)(m+1) \ .
\end{array}
\right]
$$

Every possible decimal from "$\frac{1}{10}$ of the correct product", on down through $\frac{1}{10^2}$, $\frac{1}{10^3}$, and so on, occurs if either of the lower bounds n, m is 0.

Every possible decimal of the form "10^j times the correct product" can occur, for all $j = 1, 2, ..., k$, if either

$$
\begin{aligned}
n &= 0 \text{ and } 0 \leq \varepsilon \leq \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right) \\
&\text{or} \\
m &= 0 \text{ and } 0 \leq \delta \leq \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) .
\end{aligned}
$$

The proofs of these facts are contained in the Theorem on the following pages.

In view of all this, to limit yourself to a single possible decimal, just make sure both $n \neq 0$ and $m \neq 0$. That's all!

Theorem: As notation, let $n, m \in \{0, 1, 2, ...\}$, $k \in \{1, 2, ...\}$ and δ, ε satisfy $0 \leq \delta, \varepsilon \leq 1$. Assume we have the following decimal multiplication question, with upper bound and lower bound approximations as shown:

$$\left[\begin{array}{c} n \leq n+\varepsilon \leq n+1 \\ m \leq m+\delta \leq m+1 \\ \text{Multiplying vertically,} \\ nm \leq (n+\varepsilon)(m+\delta) \leq (n+1)(m+1) \ \ . \end{array}\right]$$

Part (1)

$$\left[\begin{array}{c} \text{We have } 10^{-k} \text{ of the correct product} \\ \text{lying between the approximations, meaning} \\ nm \leq 10^{-k}(n+\varepsilon)(m+\delta) \leq (n+1)(m+1) \end{array}\right]$$

$$\Longleftrightarrow$$

$$[\ n = 0 \quad \text{or} \quad m = 0\ ].$$

Part (2)

$$\left[\begin{array}{c} \text{We have } 10^{j} \text{ times the correct product} \\ \text{lying between the approximations} \\ (\text{ for all } j = 1, 2, ..., k\ ), \text{ meaning} \\ nm \leq 10^{j}(n+\varepsilon)(m+\delta) \leq (n+1)(m+1) \end{array}\right]$$

$$\Longleftrightarrow$$

$$\left[\begin{array}{c} n = 0 \ \text{ and } \ 0 \leq \varepsilon \leq \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right) \\ \text{or} \\ m = 0 \ \text{ and } \ 0 \leq \delta \leq \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) \end{array}\right]$$

Note: To create convenient class examples, choose either

$$\begin{array}{rcl} n & = & 0 \ \text{ and } \ 0 \leq \varepsilon \leq \dfrac{1}{10^k} \\ & \text{or} & \\ m & = & 0 \ \text{ and } \ 0 \leq \delta \leq \dfrac{1}{10^k} \ . \end{array}$$

PROOF:

We note the correct product between upper bound and lower bound products:

$$\begin{aligned} nm &\leq (n+\varepsilon)(m+\delta) \leq (n+1)(m+1) \\ nm &\leq nm + n\delta + m\varepsilon + \delta\varepsilon \leq (n+1)(m+1) \ . \end{aligned}$$

Part (1): If 10^{-k} of the correct product is to lie between the two approximations, it is clear that we need only check the lefthand inequality, meaning

$$\begin{aligned} nm &\leq 10^{-k}(nm + n\delta + m\varepsilon + \delta\varepsilon) \\ \left(1 - 10^{-k}\right) nm &\leq 10^{-k}(n\delta + m\varepsilon + \delta\varepsilon) \\ \left(10^{k} - 1\right) nm &\leq n\delta + m\varepsilon + \delta\varepsilon \qquad (*) \end{aligned}$$

[$\Longleftarrow$] The statement is true if $n = 0$ because then inequality $(*)$ becomes

$$0 \leq m\varepsilon + \delta\varepsilon$$

which is true. Likewise, if $m = 0$ the associated inequality $(*)$

$$0 \leq n\delta + \delta\varepsilon$$

is similarly true.

[$\Longrightarrow$] By contrapositive, now suppose both $n \neq 0$ and $m \neq 0$, meaning both $n \geq 1$ and $m \geq 1$. We then have

$$\begin{aligned} nm &\geq n \cdot 1 \geq n\delta \ , \\ nm &\geq m \cdot 1 \geq m\varepsilon \ , \text{ and} \\ nm &\geq 1 \cdot 1 \geq \delta\varepsilon \ . \end{aligned}$$

Thus, adding vertically, we get

$$3nm \geq n\delta + m\varepsilon + \delta\varepsilon \ .$$

But then

$$\left(10^{k} - 1\right) nm > 3nm \geq n\delta + m\varepsilon + \delta\varepsilon$$

contradicting inequality $(*)$.

Part (2): If 10^k times the correct product is to lie between the two approximations (and likewise for each 10^j times the correct product for all $j = 1, 2, ..., k$), it is clear that we need only check the righthand inequality, meaning

$$\begin{aligned}
10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) &\leq (n+1)(m+1) \\
10^k(nm + n\delta + m\varepsilon + \delta\varepsilon) &\leq nm + n + m + 1 \\
\left(10^k - 1\right)nm &\leq n + m + 1 - 10^k(n\delta + m\varepsilon + \delta\varepsilon) \qquad (**)
\end{aligned}$$

[$\Longleftarrow$] If

$$n = 0 \ \text{ and } \ 0 \leq \varepsilon \leq \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)$$

then inequality $(**)$ becomes

$$\begin{aligned}
0 &\leq m + 1 - 10^k(m\varepsilon + \delta\varepsilon) \\
&= m + 1 - (m+\delta)\,10^k\varepsilon \\
&\leq m + 1 - (m+\delta)\left(\frac{m+1}{m+\delta}\right) \\
&= 0
\end{aligned}$$

which is true. Similarly, if

$$m = 0 \ \text{ and } \ 0 \leq \delta \leq \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)$$

inequality $(**)$ becomes

$$\begin{aligned}
0 &\leq n + 1 - 10^k(n\delta + \delta\varepsilon) \\
&= n + 1 - (n+\varepsilon)\,10^k\delta \\
&\leq n + 1 - (n+\varepsilon)\left(\frac{n+1}{n+\varepsilon}\right) \\
&= 0
\end{aligned}$$

which is also true.

[$\Longrightarrow$] By contrapositive, now suppose both

$$\left[\begin{array}{c} n \neq 0 \quad \text{or} \quad \varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right) \\ \text{and} \\ m \neq 0 \quad \text{or} \quad \delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) \end{array}\right]$$

Rewrite inequality $(**)$ as

$$\left(10^k - 1\right) nm + 10^k \left(n\delta + m\varepsilon + \delta\varepsilon\right) \leq n + m + 1 \ .$$

There are now 4 cases to examine:

Case 1: $n \neq 0$ and $m \neq 0$.
Since δ, ε are non-negative, we use inequality $(**)$ to see

$$\begin{aligned} n + m + 1 &\leq mn + mn + mn = 3mn < \left(10^k - 1\right) nm \\ &\leq \left(10^k - 1\right) nm + 10^k \left(n\delta + m\varepsilon + \delta\varepsilon\right) \leq n + m + 1 \ . \end{aligned}$$

But this is a contradiction. So, inequality $(**)$ cannot hold, as desired.

Case 2: $n \neq 0$ and $\delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)$.
First we rewrite the given δ inequality as

$$\begin{aligned} \delta &> \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right) \\ 10^k \delta \left(n + \varepsilon\right) &> n + 1 \\ 10^k \left(n\delta + \delta\varepsilon\right) &> n + 1 \ . \end{aligned}$$

Since ε and m are non-negative, we use this last inequality along with inequality $(**)$ to see

$$\begin{aligned} \left(10^k - 1\right) nm + 10^k m\varepsilon + n + 1 &< \left(10^k - 1\right) nm + 10^k \left(n\delta + m\varepsilon + \delta\varepsilon\right) \leq n + m + 1 \\ \left(10^k - 1\right) nm + 10^k m\varepsilon &< m \\ m\left(10^k - 1\right) n &\leq \left(10^k - 1\right) nm + 10^k m\varepsilon < m \\ m\left(10^k - 1\right) n &< m \end{aligned}$$

But since $n \neq 0$, this last inequality is a contradiction for all $m \geq 0$. So, inequality $(**)$ cannot hold, as desired.

Case 3: $\varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)$ and $m \neq 0$.
The proof here is exactly analogous to that of Case 2.

Case 4: $\varepsilon > \frac{1}{10^k}\left(\frac{m+1}{m+\delta}\right)$ and $\delta > \frac{1}{10^k}\left(\frac{n+1}{n+\varepsilon}\right)$.

Observe that here we can also assume that $n = m = 0$, since if not, we reduce to one of Cases 1, 2, or 3. Recall from Case 2 that we rewrote the δ inequality as

$$10^k\left(n\delta + \delta\varepsilon\right) > n + 1 \ .$$

Since $n = m = 0$, this inequality reduces to

$$10^k \delta\varepsilon > 1 \ .$$

Similarly, inequality $(**)$ reduces from

$$\left(10^k - 1\right) nm + 10^k\left(n\delta + m\varepsilon + \delta\varepsilon\right) \le n + m + 1$$

to

$$10^k \delta\varepsilon \le 1 \ .$$

Combining these two inequalities we conclude

$$1 < 10^k \delta\varepsilon \le 1$$

which is again a contradiction. So, for the last time, inequality $(**)$ cannot hold, as desired. ■